Rotated Ellipses

And Their Intersections With Lines

by

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Abstract:

This paper addresses the mathematical equations for ellipses rotated at any angle and how to calculate the intersections between ellipses and straight lines. The formulas for calculating the intersection points are derived, and methods are given for plotting these ellipses on a computer.

In addition, techniques are shown for determining tangents to (rotated) ellipses, calculating the ellipse's bounding box, and finding its foci.

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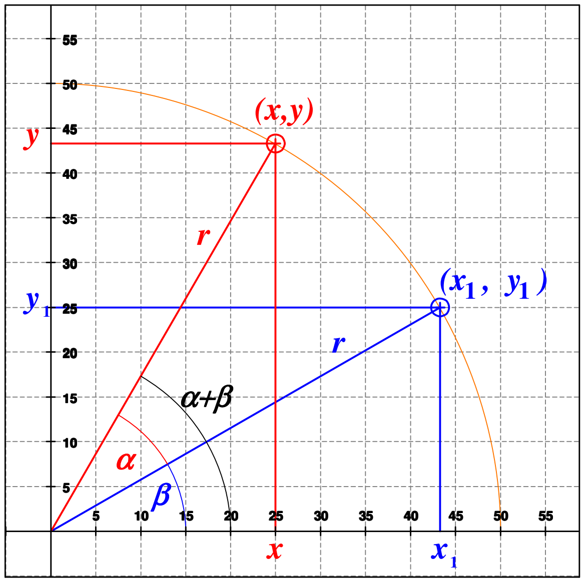
Rotating Points

First, we will rotate a point (x_1, y_1) around the origin by an angle α.



If the point (x_1, y_1) is at angle β from the x-axis, then

$$x_1 = r\cos\beta$$
$$y_1 = r\sin\beta$$

After rotating the point by angle α, the new coordinates are

$$x = r\cos(\beta + \alpha)$$
$$y = r\sin(\beta + \alpha)$$

Applying the formulae for the sine and cosine of the sum of two angles,

$$x = r\cos\beta\cos\alpha - r\sin\beta\sin\alpha = (r\cos\beta)\cos\alpha - (r\sin\beta)\sin\alpha$$
$$y = r\sin\beta\cos\alpha + r\cos\beta\sin\alpha = (r\sin\beta)\cos\alpha + (r\cos\beta)\sin\alpha$$

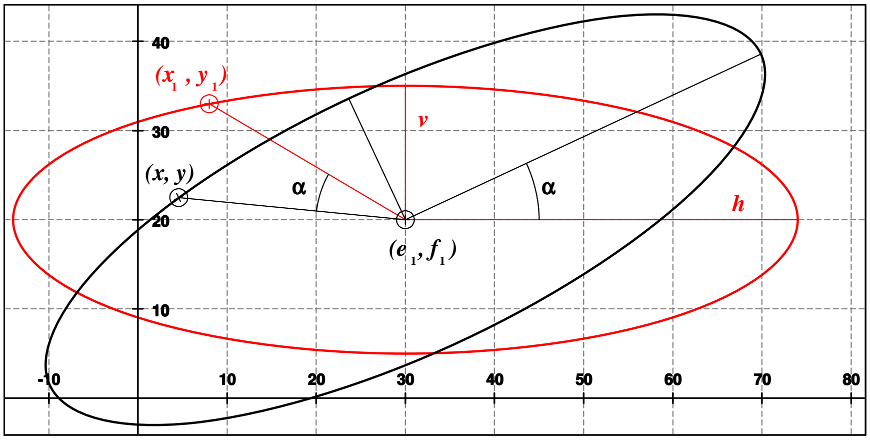
or

$$x = x_1\cos\alpha - y_1\sin\alpha \qquad (1a)$$
$$y = y_1\cos\alpha + x_1\sin\alpha \qquad (1b)$$

If you rotate that point around a "center of rotation" at (e_1, f_1), you get

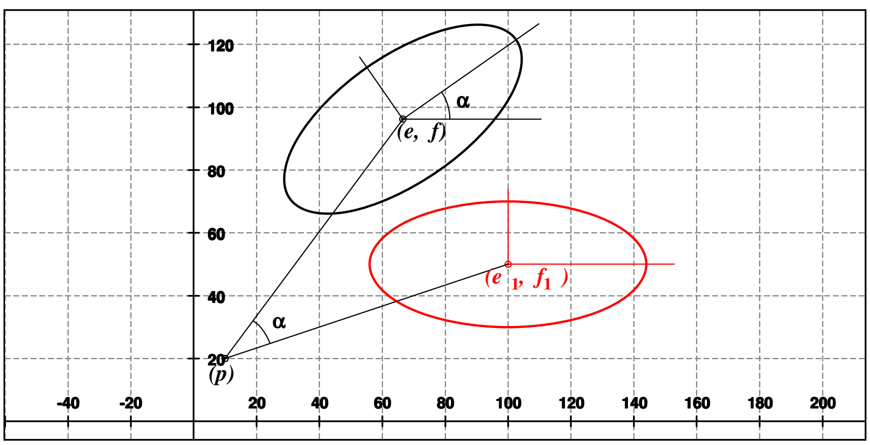
$$x = (x_1 - e_1)\cos\alpha - (y_1 - f_1)\sin\alpha + e_1 \qquad (2a)$$
$$y = (y_1 - f_1)\cos\alpha + (x_1 - e_1)\sin\alpha + f_1 \qquad (2b)$$

Rotating an Ellipse



So if (x_1, y_1) is a point on the ellipse and (e_1, f_1) is the **center of the ellipse** (see above figure), then equations (2) are true for all points on the rotated ellipse. The "line" from (e_1, f_1) to each point on the ellipse gets rotated by α.

To **rotate** an ellipse about a point (p) **other then its center**, we must rotate every point on the ellipse around point p, including the center of the ellipse.



This is as if we put a pin in the graph at point p and rotated the entire sheet of paper around the pin.

Here we are rotating the red ellipse centered at (e_1, f_1) around point (p) by an angle α.

So, for every point (x_1, y_1) on the original ellipse, the rotated point is

$$x = (x_1 - p_x)\cos\alpha - (y_1 - p_y)\sin\alpha + p_x$$
$$y = (y_1 - p_y)\cos\alpha + (x_1 - p_x)\sin\alpha + p_y$$

and the rotated center is

$$e = (e_1 - p_x)\cos\alpha - (f_1 - p_y)\sin\alpha + p_x$$
$$f = (f_1 - p_y)\cos\alpha + (e_1 - p_x)\sin\alpha + p_y$$

So the directed line from the new center (e, f) to the rotated point (x, y) can be expressed as

$$(x - e) = (x_1 - p_x - e_1 + p_x)\cos\alpha - (y_1 - p_y - f_1 + p_y)\sin\alpha$$
$$x = (x_1 - e_1)\cos\alpha - (y_1 - f_1)\sin\alpha + e$$

$$(y - f) = (y_1 - p_y - f_1 + py)\cos\alpha + (x_1 - p_x - e_1 + p_x)\sin\alpha$$
$$y = (y_1 - f_1)\cos\alpha + (x_1 - e_1)\sin\alpha + f$$

which have the same form as equations (2) for the ellipse rotated around its center, except that the new ellipse is centered at (e, f).

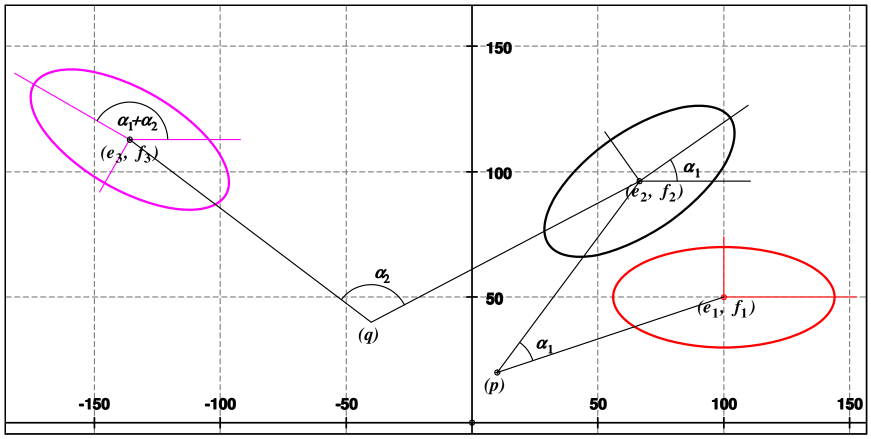
Note: If we are rotating about the center, then $(p) = (e_1, f_1)$ and $(e, f) = (e_1, f_1)$ and we are back to equations (2).

Therefore, we can state that:

When an ellipse gets rotated by angle α about a point p ***other than its center***, the **center** of the ellipse **gets rotated about point *p*** and the **new ellipse** at the new center gets **rotated about the new center** by angle α.

Rotating an Ellipse More Than Once

If an ellipse is rotated around one center of rotation and then again around a different point, what is the result?



Let us assume that the ellipse has its center at (e_1, f_1). The ellipse is first rotated about the point p by angle of α_1, and then rotated about the point q by angle α_2.

Using the rule derived in the preceding section, we can compute the center point (e_2, f_2) after the first rotation

$$
\begin{aligned}
e_2 &= (e_1 - p_x)\cos\alpha_1 - (f_1 - p_y)\sin\alpha_1 + p_x \\
f_2 &= (f_1 - p_y)\cos\alpha_1 + (e_1 - p_x)\sin\alpha_1 + p_y
\end{aligned}
$$

Then compute the final center point (e_3, f_3) with similar equations

$$
\begin{aligned}
e_3 &= (e_2 - q_x)\cos\alpha_2 - (f_2 - q_y)\sin\alpha_2 + q_x \\
f_3 &= (f_2 - q_y)\cos\alpha_2 + (e_2 - q_x)\sin\alpha_2 + q_y
\end{aligned}
$$

Finally, the ellipse is plotted centered at point (e_3, f_3) with a rotation of $(\alpha_1 + \alpha_2)$.

In other words,

> The **center point is** successively **rotated** (translated) around **each** of the two centers of rotation, and the **ellipse** itself is rotated **about the new center** by the **sum of the two rotation angles**.

This makes plotting the rotated ellipses easier, since you only have to rotate the points on the ellipse once.

This process can be applied any number of times for **multiple rotations** around multiple points.

Constructing (Plotting) an Ellipse

For a non-rotated ellipse, it is easy to show that

$$x = h\cos\beta \tag{3a}$$
$$y = v\sin\beta \tag{3b}$$

satisfies the equation $\frac{x^2}{h^2}+\frac{y^2}{v^2}=1$. Simply substitute

$$\frac{(h\cos\beta)^2}{h^2}+\frac{(v\sin\beta)^2}{v^2}=\frac{h^2\cos^2\beta}{h^2}+\frac{v^2\sin^2\beta}{v^2}=\cos^2\beta+\sin^2\beta\equiv 1\,.$$

Therefore, equations (3) satisfy the equation for a non-rotated ellipse, and you can simply plot them for all values of β from 0 to 360 degrees. For ellipses not centered at the origin, simply add the coordinates of the center point (*e, f*) to the calculated (*x, y*).

Constructing (Plotting) a Rotated Ellipse

if we let

$$x_1 = h\cos\beta$$
$$y_1 = v\sin\beta$$

and then rotate these points around the origin by angle α,

$$x = x_1\cos\alpha - y_1\sin\alpha$$
$$y = y_1\cos\alpha + x_1\sin\alpha$$

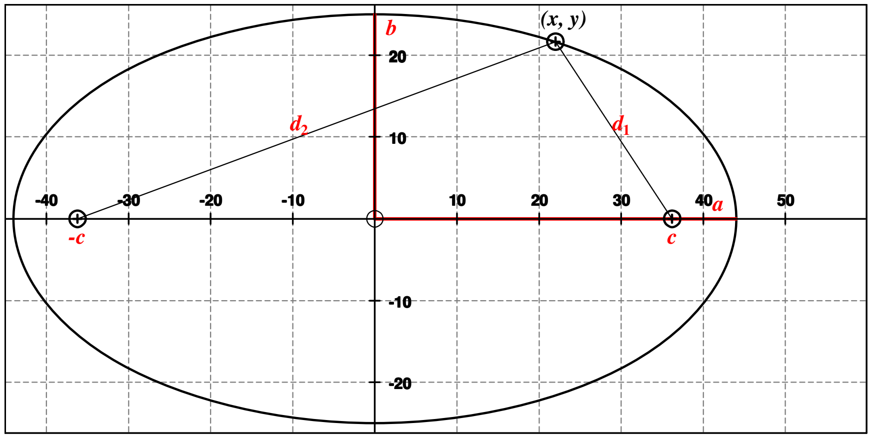
substituting

$$x = h\cos\beta\cos\alpha - v\sin\beta\sin\alpha \tag{4a}$$
$$y = v\sin\beta\cos\alpha + h\cos\beta\sin\alpha \tag{4b}$$

You can plot them for all values of β from 0 to 360 degrees. For ellipses not centered at the origin, simply add the coordinates of the center point (*e, f*) to the calculated (*x, y*).

If the ellipse is rotated multiple times around multiple points, first calculate the new center point by successively rotating it around each center of rotation (equations 2), then plot the ellipse at the new center point, rotating the ellipse by the sum of the rotation angles.

Finding the Foci of an Ellipse



If you need to compute its foci, the conversion is easy. The formula is

$$c^2 = a^2 - b^2$$

where *a* is the major axis and *b* is the minor axis (measured from the center to the edge of the ellipse). *c* is the distance from the center to each focus. The foci lie along the major axis. If the ellipse is a circle ($a=b$), then $c=0$.

You can reverse this conversion if you know the foci and *either* of the axes, however if all you have is the foci, you cannot determine *a* and *b*.

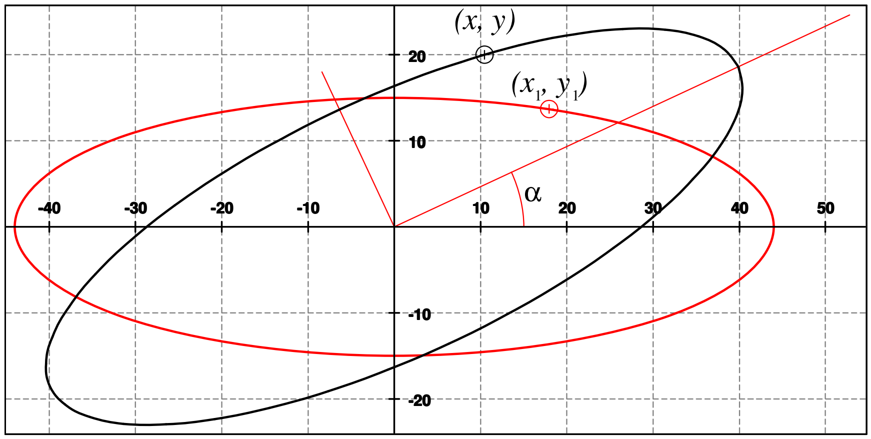
If you know the foci ***and*** any point (*x, y*) on the ellipse, you can calculate the sum of the distances to the two foci:

$$d_1 = \sqrt{(x-c)^2 + y^2}$$
$$d_2 = \sqrt{(x+c)^2 + y^2}$$

For any point on the ellipse, $d_1 + d_2 = 2a$. Then you can calculate $b^2 = a^2 - c^2$.

Intersection of Lines with a Rotated Ellipse

Assume we have an ellipse with horizontal radius h and vertical radius v, centered at the origin (for now), and rotated counter-clockwise by angle α.



The equation for the non-rotated (red) ellipse is

$$\frac{x_1^{\,2}}{h^2} + \frac{y_1^{\,2}}{v^2} = 1 \qquad (5)$$

where x_1 and y_1 are the coordinates of points on the ellipse rotated back (clockwise) by angle α to produce a "regular" ellipse, with the axes of the ellipse parallel to the x and y axes of the graph ("red" ellipse).

Using the equations for rotating a point about the origin by angle α <u>clockwise</u> (that is, a counter-clockwise rotation of $-\alpha$) , we get

$$x_1 = x\cos(-\alpha) - y\sin(-\alpha) \quad \text{and} \quad y_1 = y\cos(-\alpha) + x\sin(-\alpha)$$

or

$$x_1 = x\cos(\alpha) + y\sin(\alpha) \qquad \text{and} \quad y_1 = y\cos(\alpha) - x\sin(\alpha)$$

Plugging these into equation (5) for the non-rotated ellipse, we get the equation for a **<u>rotated</u>** ellipse:

$$\frac{(x\cos\alpha + y\sin\alpha)^2}{h^2} + \frac{(y\cos\alpha - x\sin\alpha)^2}{v^2} = 1 \qquad (6)$$

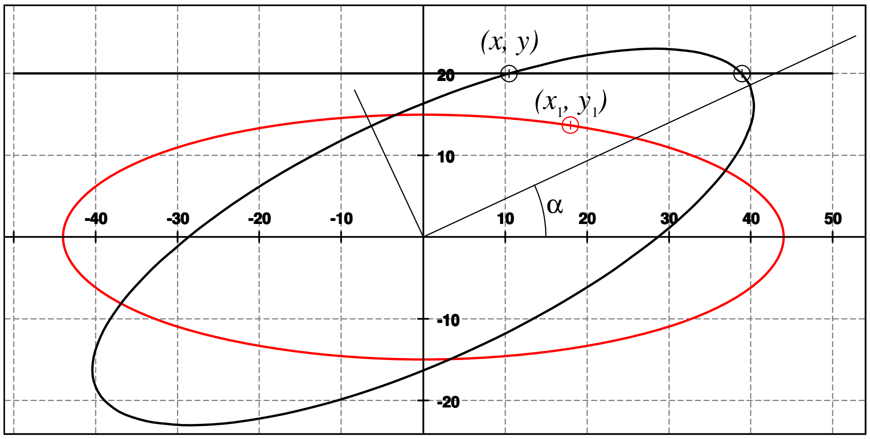
Expanding,

$$\frac{x^2\cos^2\alpha + 2xy\cos\alpha\sin\alpha + y^2\sin^2\alpha}{h^2} + \frac{y^2\cos^2\alpha - 2xy\cos\alpha\sin\alpha + x^2\sin^2\alpha}{v^2} = 1$$

or

$$v^2x^2\cos^2\alpha + 2v^2xy\cos\alpha\sin\alpha + v^2y^2\sin^2\alpha$$
$$+h^2y^2\cos^2\alpha - 2h^2xy\cos\alpha\sin\alpha + h^2x^2\sin^2\alpha - h^2v^2 = 0 \qquad (7)$$

Intersection with a Horizontal Line.



Assume that y is a constant, and re-write equation (7) as a quadratic equation for x:

$$\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right)x^2 + 2y\cos\alpha\sin\alpha\left(v^2 - h^2\right)x$$
$$+ y^2\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right) - h^2v^2 = 0 \qquad (8)$$

So we get the parameters for the quadratic equation solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \text{where:}$$

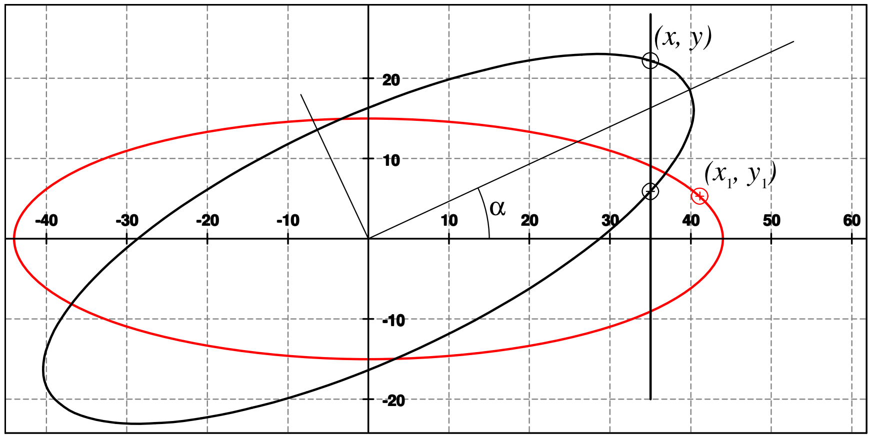
$$a = v^2\cos^2\alpha + h^2\sin^2\alpha \qquad (9a)$$
$$b = 2y\cos\alpha\sin\alpha\left(v^2 - h^2\right) \qquad (9b)$$
$$c = y^2\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right) - h^2v^2 \qquad (9c)$$

Now, returning to the general case where the ellipse is not centered at the origin, assume the center of the ellipse is at (e, f). The translation is simple. Merely subtract f from the value of y, and add e to the resulting values of x.

Note: For a "normal" or non-rotated ellipse, ($\alpha = 0$), the equations simplify to:

$$a = v^2 \qquad (10a)$$
$$b = 0 \qquad (10b)$$
$$c = h^2\left(y^2 - v^2\right) \qquad (10c)$$

Intersection with a Vertical Line.



Assume that x is a constant, re-write equation (7) as a quadratic equation for y:

$$\left(v^2 \sin^2 \alpha + h^2 \cos^2 \alpha\right) y^2 + 2x \cos\alpha \sin\alpha \left(v^2 - h^2\right) y$$

$$+ x^2\left(v^2 \cos^2 \alpha + h^2 \sin^2 \alpha\right) - h^2 v^2 = 0 \qquad (11)$$

So we get the parameters for the quadratic equation solution **for *y***

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \text{where:}$$

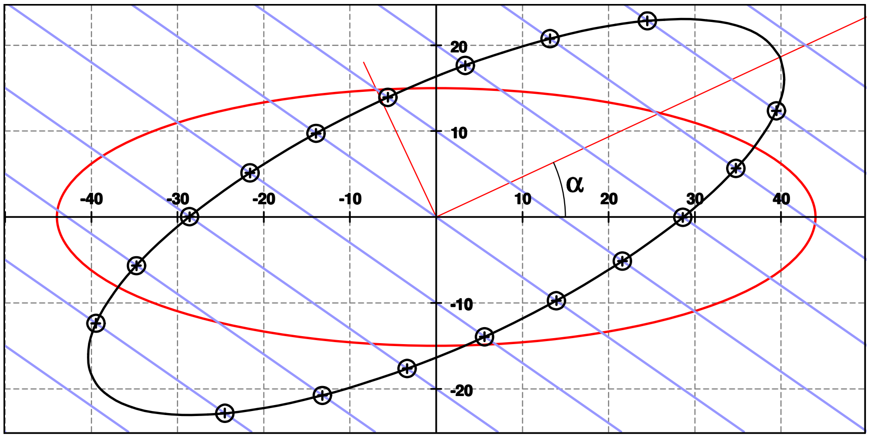
$$a = v^2 \sin^2 \alpha + h^2 \cos^2 \alpha \qquad (12a)$$
$$b = 2x \cos\alpha \sin\alpha \left(v^2 - h^2\right) \qquad (12b)$$
$$c = x^2\left(v^2 \cos^2 \alpha + h^2 \sin^2 \alpha\right) - h^2 v^2 \qquad (12c)$$

Now, returning to the general case where the ellipse is not centered at the origin, assume the center of the ellipse is at (e, f). The translation is simple. Merely subtract e from the value of x, and add f to the resulting values of y.

Note: For a "normal" or non-rotated ellipse, ($\alpha = 0$), the equations simplify to:

$$a = h^2 \qquad (13a)$$
$$b = 0 \qquad (13b)$$
$$c = v^2\left(x^2 - h^2\right) \qquad (13c)$$

Intersection of Rotated Ellipse with Sloping Line(s)



This is the general case where the ellipse is rotated by angle α and the line has the equation

$$y = mx + b_1 \qquad \text{where} \quad m \neq \infty \text{, } e.g.\text{, the line is } \textbf{\underline{not vertical}}.$$

Note: We use b_1 instead of b to avoid confusion with the b parameter in the quadratic equation.

Substituting this expression for y in equation (7),

$$\begin{aligned} & v^2x^2\cos^2\alpha + 2v^2x(mx+b_1)\cos\alpha\sin\alpha + v^2(mx+b_1)^2\sin^2\alpha \\ & \quad + h^2(mx+b_1)^2\cos^2\alpha - 2h^2x(mx+b_1)\cos\alpha\sin\alpha + h^2x^2\sin^2\alpha - h^2v^2 = 0 \end{aligned}$$

Expanding,

$$\begin{aligned} & v^2x^2\cos^2\alpha + 2v^2mx^2\cos\alpha\sin\alpha + 2v^2xb_1\cos\alpha\sin\alpha \\ & \quad + v^2m^2x^2\sin^2\alpha + 2v^2mxb_1\sin^2\alpha + v^2b_1^{\,2}\sin^2\alpha \\ & \quad + h^2m^2x^2\cos^2\alpha + 2h^2mxb_1\cos^2\alpha + h^2b_1^{\,2}\cos^2\alpha \\ & \quad - 2h^2mx^2\cos\alpha\sin\alpha - 2h^2xb_1\cos\alpha\sin\alpha + h^2x^2\sin^2\alpha - h^2v^2 = 0 \end{aligned}$$

Collecting terms, as before,

$$\begin{aligned} a &= v^2\cos^2\alpha + 2v^2m\cos\alpha\sin\alpha + v^2m^2\sin^2\alpha + h^2m^2\cos^2\alpha \\ & \quad - 2h^2m\cos\alpha\sin\alpha + h^2\sin^2\alpha \\ b &= 2v^2b_1\cos\alpha\sin\alpha + 2v^2mb_1\sin^2\alpha + 2h^2mb_1\cos^2\alpha - 2h^2b_1\cos\alpha\sin\alpha \\ c &= v^2b_1^{\,2}\sin^2\alpha + h^2b_1^{\,2}\cos^2\alpha - h^2v^2 \end{aligned}$$

or

$$a = v^2\left(\cos^2\alpha + 2m\cos\alpha\sin\alpha + m^2\sin^2\alpha\right) + h^2\left(m^2\cos^2\alpha - 2m\cos\alpha\sin\alpha + \sin^2\alpha\right) \quad (14a)$$

$$b = 2v^2 b_1\left(\cos\alpha\sin\alpha + m\sin^2\alpha\right) + 2h^2 b_1\left(m\cos^2\alpha - \cos\alpha\sin\alpha\right) \quad (14b)$$

$$c = b_1^{\,2}\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right) - h^2v^2 \quad (14c)$$

where, again,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Once the value(s) for x is calculated, use the equation $y = mx + b_1$ (15)
to calculate the corresponding y values for the intersection points.

<u>Note:</u> This method **<u>will not work for vertical lines</u>** (infinite slope). That would require reversing the above solution to solve for *y* instead of *x*. It is much easier, in this case, to use the special formula described above for intersections with vertical lines (equations (12)).

<u>Note:</u> Substituting $m = 0$ and b_1 = y (for a horizontal line) into the above formulas for *a*, *b*, and *c* (equations 14), we get the same formulas as equations (9).

Again, returning to the **general case** where the ellipse is **<u>not centered at the origin</u>**, assume the center of the ellipse is at (e, f). The equation for the line must also be translated to coordinates with their origin at (e, f). Letting x_1 and y_1 be those coordinates,

$$x_1 = x - e \qquad\qquad y_1 = y - f$$

and the equation for the line becomes:

$$y_1 + f = m(x_1 + e) + b_1$$

or

$$y_1 = mx_1 + (me + b_1 - f).$$

So, in the new coordinate system,

$$y_1 = mx_1 + b_2\,, \text{ where } \quad b_2 = b_1 + me - f$$

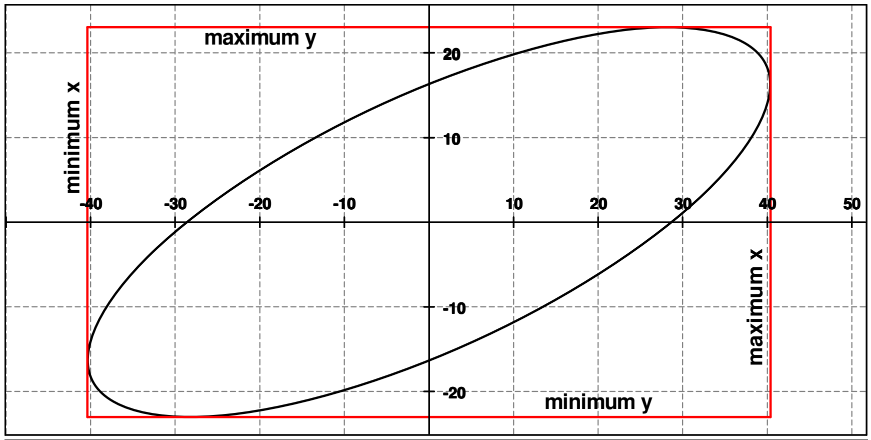
Therefore, we must add $(me - f)$ to b_1 in equations (14). Then calculate the intersection points, and add *e* to each calculated *x*-value to return to the original coordinate system. Finally, calculate *y* for each *x* using $y = mx + b_1$ (equation (15)), where b_1 is the original value.

In the special case, for non-rotated ellipses ($\alpha = 0$) with sloping line ($m \neq 0$):

$$a = v^2 + h^2 m^2 \qquad (11a)$$
$$b = 2h^2 b_1 m \qquad (11b)$$
$$c = h^2 \left(b_1{}^2 - v^2\right) \qquad (11c)$$

And for the horizontal line ($m = 0$, $b_1 = y$), the equations further reduce to the same as equations (10) for non-rotated ellipses.

Bounding Box for a Rotated Ellipse



We would like to know the maximum and minimum horizontal and vertical values for a rotated ellipse, *i.e.*, the **bounding box** for the ellipse.

Assume, as before, that the ellipse is centered at the origin. The top line of the bounding box, at $y=y_{max}$, may be thought of as a horizontal line which intersects the ellipse at a single point. Equations (9) give the a, b and c values for solving the quadratic equation for x on that line:

$$a = v^2\cos^2\alpha + h^2\sin^2\alpha \qquad (8a)$$
$$b = 2y\cos\alpha\sin\alpha\left(v^2 - h^2\right) \qquad (9b)$$
$$c = y^2\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right) - h^2v^2 \qquad (9c)$$

Note that a does not contain the variable y, therefore a is a constant for this condition.

Recalling that you get a single solution when

$$b^2 - 4ac = 0 \qquad (17)$$

we can plug in the values for a, b and c from equations (9) and solve for y.

$$\left(2y\cos\alpha\sin\alpha\left(v^2 - h^2\right)\right)^2 - 4a\left(y^2\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right) - h^2v^2\right) = 0$$

Separate out the y terms:

$$\left(2\cos\alpha\sin\alpha\left(v^2 - h^2\right)\right)^2 y^2 - 4a\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right)y^2 + 4ah^2v^2 = 0$$

$$\left(4\cos^2\alpha\sin^2\alpha\left(v^2 - h^2\right)^2 - 4a\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right)\right)y^2 + 4ah^2v^2 = 0$$

$$y^2 = \frac{-4ah^2v^2}{4\cos^2\alpha\sin^2\alpha\left(v^2 - h^2\right)^2 - 4a\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right)}$$

$$y = \pm\sqrt{\frac{-ah^2v^2}{\cos^2\alpha\sin^2\alpha\left(v^2-h^2\right)^2 - a\left(v^2\sin^2\alpha + h^2\cos^2\alpha\right)}} \qquad (18)$$

where $\qquad a = v^2\cos^2\alpha + h^2\sin^2\alpha \qquad$ (9a)

The positive and negative values give solutions for y_{max} and y_{min}. There is no need to find the x-values corresponding to these y-values at this time. If the ellipse is not centered at the origin, add the vertical offset of the ellipse center.

Turning now to the maximum and minimum ***horizontal*** values, go back to the intersection of a ***vertical*** line with the ellipse. The right line of the bounding box, at $x = x_{max}$, may be thought of as a vertical line which intersects the ellipse at a single point. Equations (12) give the a, b and c values for solving the quadratic equation for y on that line:

$$a = v^2\sin^2\alpha + h^2\cos^2\alpha \qquad (12a)$$
$$b = 2x\cos\alpha\sin\alpha\left(v^2-h^2\right) \qquad (12b)$$
$$c = x^2\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right) - h^2v^2 \qquad (12c)$$

and letting $\quad b^2 - 4ac = 0$, as before, solve for x:

$$\left(2x\cos\alpha\sin\alpha\left(v^2-h^2\right)\right)^2 - 4a\left(x^2\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right) - h^2v^2\right) = 0$$

Separate out the x terms:

$$\left(2\cos\alpha\sin\alpha\left(v^2-h^2\right)\right)^2 x^2 - 4a\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right)x^2 + 4ah^2v^2 = 0$$

$$\left(4\cos^2\alpha\sin^2\alpha\left(v^2-h^2\right)^2 - 4a\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right)\right)x^2 + 4ah^2v^2 = 0$$

$$x^2 = \frac{-ah^2v^2}{\cos^2\alpha\sin^2\alpha\left(v^2-h^2\right)^2 - a\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right)}$$

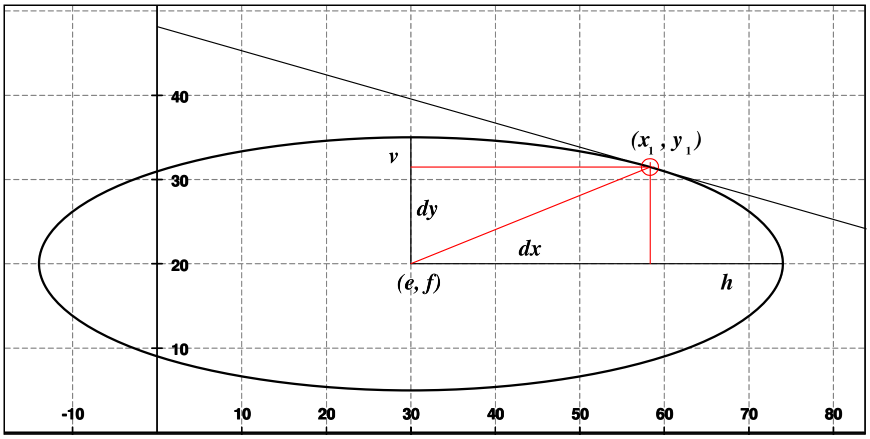
$$x = \pm\sqrt{\frac{-ah^2v^2}{\cos^2\alpha\sin^2\alpha\left(v^2-h^2\right)^2 - a\left(v^2\cos^2\alpha + h^2\sin^2\alpha\right)}} \qquad (19)$$

where $\qquad a = v^2\sin^2\alpha + h^2\cos^2\alpha \qquad$ (12a)

The positive and negative values give solutions for x_{max} and x_{min}. There is no need to find the y-values corresponding to these x-values at this time. If the ellipse is not centered at the origin, add the horizontal offset of the ellipse center.

Determining the Tangent to an Ellipse

Assume we have an ellipse with horizontal radius h and vertical radius v centered at (e, f). We wish to determine the tangent to the ellipse at point (x_1, y_1).



If this were a circle, the tangent would be negative inverse of the slope (dy/dx) of the radial line, or $-dx/dy$. However, because the axes are not equal, we have to make them equal by first multiplying dy/dx by h/v. Then take the negative inverse = $(-dx/dy)(v/h)$. This gives the slope if the ellipse were a circle, so we now have to multiply again by v/h to re-proportion the circle back to an ellipse.

Thus, the slope m_1 of the tangent can be calculated as

$$m_1 = -\frac{dx}{dy} \times \frac{v^2}{h^2} = -\frac{x_1 - e}{y_1 - f} \times \frac{v^2}{h^2} \qquad (20)$$

and the equation for the tangent line is

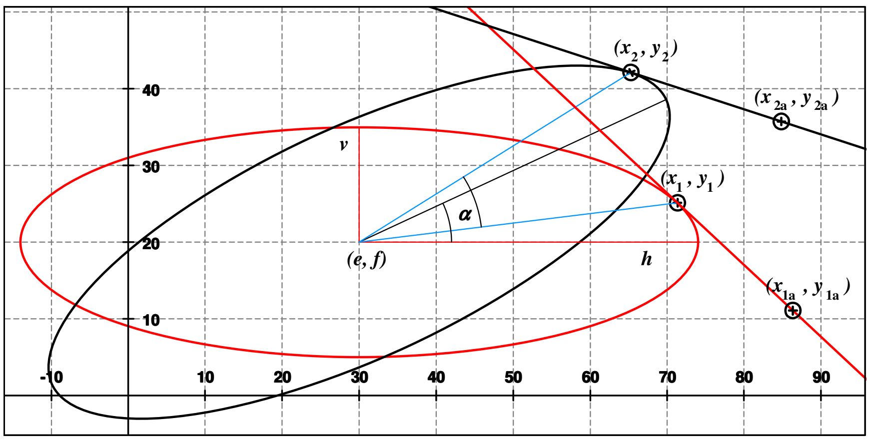
$$y = m_1(x - x_1) + y_1 \qquad (21)$$

Note: if $y_1 = f$, the slope is infinite and the equation is $x = x_1$ for all y.

Determining the Tangent to a Rotated Ellipse

Assume the ellipse in question is rotated by angle α, and we wish to find the tangent line at (x_2, y_2) . First rotate the (x_2, y_2) point back to the corresponding point on a "normal" or un-rotated ellipse with the equations

$$x_1 = (x_2 - e)\cos\alpha + (y_2 - f)\sin\alpha + e \tag{22a}$$
$$y_1 = (y_2 - f)\cos\alpha - (x_2 - e)\sin\alpha + f\ . \tag{22b}$$



Now we can solve for the tangent line of the un-rotated ellipse, using the technique from the previous section. Substituting x_1 and y_1 from equations (22) into equation (20),

$$m_1 = -\frac{(x_2 - e)\cos\alpha + (y_2 - f)\sin\alpha}{(y_2 - f)\cos\alpha - (x_2 - e)\sin\alpha} \times \frac{v^2}{h^2} \tag{23}$$

which is the slope of the tangent to the un-rotated ellipse in terms of the rotated points (x_2, y_2).

Since the ellipse was rotated, we must rotate the tangent line back to the angle of the rotated ellipse. In order to rotate the line, we need more than one point on the line.

First, we will work with the case where ***m_1* is not infinite**.

We will pick an arbitrary point (x_{1a}, y_{1a}) on the tangent line.

Let $\qquad x_{1a} = x_1 + z \quad$ where $z > 0$,

then $\qquad y_{1a} = m_1(x_{1a} - x_1) + y_1 \qquad$ (from equation 2)

$$= m_1(x_1 + z - x_1) + y_1$$
$$= m_1 z + y_1$$

Now let us rotate (x_{1a}, y_{1a}) back to the rotated ellipse. Call this rotated point (x_{2a}, y_{2a})

$$
\begin{aligned}
x_{2a} &= (x_{1a} - e)\cos\alpha - (y_{1a} - f)\sin\alpha + e \\
&= (x_1 + z - e)\cos\alpha - (m_1 z + y_1 - f)\sin\alpha + e \qquad (24a) \\
y_{2a} &= (y_{1a} - f)\cos\alpha + (x_{1a} - e)\sin\alpha + f \\
&= (m_1 z + y_1 - f)\cos\alpha + (x_1 + z - e)\sin\alpha + f \qquad (24b)
\end{aligned}
$$

Now the slope m_2 of the rotated tangent line at (x_2, y_2) can be expressed as dy_2/dx_2.

$$
\begin{aligned}
x_2 &= (x_1 - e)\cos(\alpha) - (y_2 - f)\sin(\alpha) + e \qquad (25a) \\
y_2 &= (y_1 - f)\cos(\alpha) + (x_1 - e)\sin(\alpha) + f\,. \qquad (25b)
\end{aligned}
$$

Now, using the expressions for (x_{2a}, y_{2a}) in equations (24) and the expressions for (x_2, y_2) in equations (25), we can calculate

$$
\begin{aligned}
dx_2 &= x_{2a} - x_2 \\
&= (x_1 + z - e - (x_1 - e))\cos\alpha - (m_1 z + y_1 - f - (y_1 - f))\sin\alpha + e - e \\
&= z\cos\alpha - m_1 z\sin\alpha \\
&= z(\cos\alpha - m_1\sin\alpha) \\
dy_2 &= y_{2a} - y_2 \\
&= (m_1 z + y_1 - f - (y_1 - f))\cos\alpha + (x_1 + z - e - (x_1 - e))\sin\alpha + f - f \\
&= m_1 z\cos\alpha + z\sin\alpha \\
&= z(m_1\cos\alpha + \sin\alpha)
\end{aligned}
$$

$$
m_2 = \frac{dy_2}{dx_2} = \frac{m_1\cos\alpha + \sin\alpha}{\cos\alpha - m_1\sin\alpha}\,. \qquad (26)
$$

<u>**Note**</u>: If $\cos\alpha - m_1\sin\alpha = 0$, the rotated slope is infinite.

In the case where the <u>**un-rotated slope m_1 is infinite**</u>, we can still calculate the rotated slope. In this case,

$$x_{1a} = x_1 \qquad (27a)$$

and let $\quad y_{1a} = y_1 + z$, where $z > 0$. $\qquad$ (27b)

Repeating the above calculations for the new (x_a, y_a) ,

$$
\begin{aligned}
x_{2a} &= (x_{1a} - e)\cos\alpha - (y_{1a} - f)\sin\alpha + e \\
&= (x_1 - e)\cos\alpha - (y_1 + z - f)\sin\alpha + e \qquad (28a) \\
y_{2a} &= (y_{1a} - f)\cos\alpha + (x_{1a} - e)\sin\alpha + f \\
&= (y_1 + z - f)\cos\alpha + (x_1 - e)\sin\alpha + f \qquad (28b)
\end{aligned}
$$

Now, using the expressions for (x_{2a}, y_{2a}) in equations (28) and the expressions for (x_2, y_2) in equations (25), we can calculate

$$
\begin{aligned}
dx_2 &= x_{2a} - x_2 \\
&= (x_1 - e - (x_1 - e))\cos\alpha - (y_1 + z - f - (y_1 - f))\sin\alpha + e - e \\
&= -z\sin\alpha \\
dy_2 &= y_{2a} - y_2 \\
&= (y_1 + z - f - (y_1 - f))\cos\alpha + (x_1 - e - (x_1 - e))\sin\alpha + f - f \\
&= z\cos\alpha
\end{aligned}
$$

$$
m_2 = \frac{dy_2}{dx_2} = \frac{\cos\alpha}{-\sin\alpha}, \qquad (29)
$$

which, incidentally, is the limit of equation (26) as $m_1 \to \infty$.

Note In equation (29), m_2 is infinite if $\sin\alpha = 0$, that is, rotating a line of infinite slope by a multiple of 180 degrees ($\sin\alpha = 0$) results in another infinite slope.

Summary: The slope m_2 of the tangent at a point (x_2, y_2) on an ellipse rotated by angle α is determined by first calculating the tangent m_1 on the un-rotated ellipse using equation (23). Then, if m_1 is infinite, calculate m_2 using equation (29), otherwise use equation (26).

In either case, the equation for the tangent line is

$$
y = m_2(x - x_2) + y_2 \qquad (30)
$$

Approximating a Segment of an Ellipse with a Bezier Curve

If you are drawing an elliptical arc, chord, or pie segment, it can be useful (and often faster) to draw the segment as a cubic Bezier curve. This works for small segments, not more than 90 degrees and optimally, less than 45 degrees. The smaller the segment, the greater the accuracy. The larger you magnify a segment of an ellipse, the straighter the curve appears.

It is assumed that you know the horizontal and vertical radii of the ellipse (*h* and *v*), the center of the ellipse (*e, f*) and the rotation angle α, if the ellipse is rotated.

If the ellipse is rotated, we will first rotate it back to a "normal" un-rotated ellipse. If you are starting with 2 points (x_{start}, y_{start}) and (x_{end}, y_{end}) on the rotated ellipse, rotate each of them *back* using the formula

$$x_1 = (x-e)\cos(\alpha) + (y-f)\sin(\alpha) + e \qquad \text{and}$$
$$y_1 = (y-f)\cos(\alpha) - (x-e)\sin(\alpha) + f\ .$$

Next determine the start and end angles. To do that, we first define a function

$$\arctan 2(x,y) = \left\{ \begin{array}{c} \arctan\dfrac{y}{x},\ \text{ if } x>0,\ y\ge 0 \\ \pi - \arctan\dfrac{y}{-x},\ \text{ if } x<0,\ y\ge 0 \\ \pi + \arctan\dfrac{y}{x},\ \text{ if } x<0,\ y<0 \\ 2\pi - \arctan\dfrac{-y}{x},\ \text{ if } x>0,\ y<0 \\ \pi/2,\ \text{ if } x=0,\ y\ge 0 \\ 3\pi/2,\ \text{ if } x=0,\ y<0 \end{array} \right\}, \text{ where the angle is in radians.}$$

This will result in a single value between 0 and 2π (0 and 360 degrees).

These start and end angles, β_{start} and β_{end}, are the angles used in the formulae

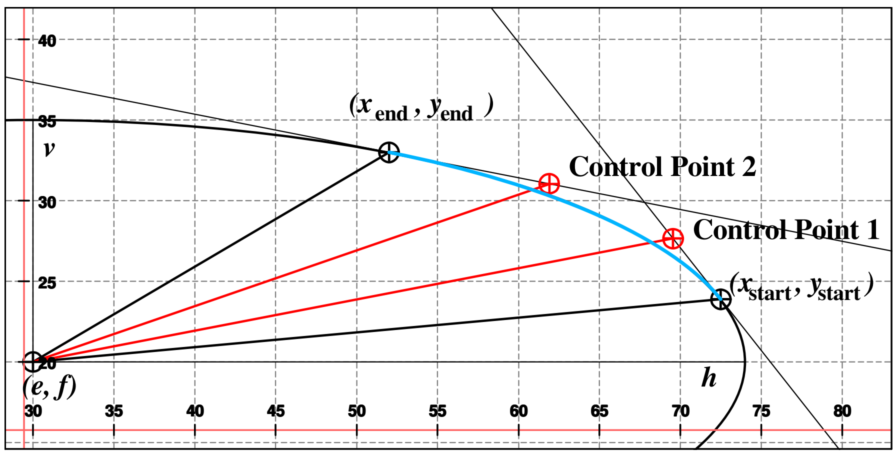
$$x_1 - e = h\cos\beta$$
$$y_1 - f = v\sin\beta$$

defining the ellipse. They are not actual angles unless the ellipse is a circle. To find the angles, use the arctan2 formula, above, as follows:

$$\beta = \arctan 2\left(\frac{x_1-e}{h}, \frac{y_1-f}{v}\right)$$

To determine the Bezier control points, assume they will lie on lines tangent to the ellipse at the start and end points. Determine the parameters for the tangent lines at both ends of the segment, using the technique described in the previous section. Now calculate angles of approximately 1/3 and 2/3 of the angle (β_{end} - β_{start}) and spanned by the segment. Empirically, the values .32343333 and .67636667 work quite well. Adjust the angles if the arc crosses 360 degrees.

Now imagine lines from the center of the ellipse at those angles and find their intersection with the tangent lines. (Take into account the possibility that the tangent or radial lines may have infinite slope.) These intersections are your Bezier control points.



Now, if the ellipse is rotated, rotate the Bezier control points by the same angle, using the formulae

$$x = (x_1 - e)\cos\alpha - (y_1 - f)\sin\alpha + e$$
$$y = (y_1 - f)\cos\alpha + (x_1 - e)\sin\alpha + f\ .$$

Now draw the Bezier curve from the (rotated) start point to the (rotated) end point using the calculated (rotated) control points. The curve is shown in blue on the figure above.